2. ALEKSANDROV V.M. and MKHITARYAN S.M., Contact Problems for Bodies with Thin Coatings and Interlayers, Nauka, Moscow, 1983.
3. BELOTSERKOVSKII S.M. and LIFANOV I.K., Numerical Methods in Singular Integral Equations, Nauka, Moscow, 1985.
4. NOWACKI W., Theory of Elasticity, Mir, Moscow, 1975.

Translated by M.D.F.

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ON TWO MIXED PROBLEMS OF ANTIPLANE STRAIN OF AN ELASTIC WEDGE WITH CIRCULAR HOLES*

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Two problems are examined: 1) a wedge with a circular hole, clamped along the lower face, is subjected to the action of shear forces along the upper face, and 2) a rigid stamp acts on the upper face instead of shear forces. The circular hole is assumed to be load-free. Both problems reduce to a set of infinite systems of linear equations with a completely continuous operator l_2 under the condition that the circle does not touch the sides of the angle. These equations enable the method of reduction to be used. Formulas obtained, that relate the basis solutions of the Laplace equation in two different polar coordinate systems, are utilized in the solution. The method can be extended to the case of a wedge with several circular holes.

The problem of the deformation of a wedge with a circular hole was first examined in one special in /1/, however, the infinite system obtained there remained uninvestigated.

1. We present the relationships between the basis solutions of Laplace's equation in a plane (Figs.1 and 2; $OO_1 = h$, $O_2O_2 = R$), which enable us to change from one system of polar coordinates to another

$$\rho^{-s}e^{is\varphi} = \left(\frac{e^{i\alpha}}{h}\right)^s \Gamma(1-s) \sum_{n=0}^{\infty} \left(\frac{\rho_1}{h}\right)^n \frac{e^{-in\varphi_1}}{n!\,\Gamma(1-s-n)} \quad (\rho_1 < h) \tag{1.1}$$

$$\left(\frac{\rho_1}{h}\right)^{-n} e^{\pm in\omega_{1,2}} = \frac{i}{2(n-1)!} \int_{\Gamma} \frac{\Gamma(s)\, h^s \rho^{-s}}{\sin \pi s\, \Gamma(1+s-n)} e^{\pm is\psi_{1,2}}\, ds \tag{1.2}$$

$$(\Gamma\colon 0 < \operatorname{Re} s < 1, \quad s = \alpha + i\tau), \quad 0 \leqslant \varphi_1 < 2\pi$$
$$\psi_1 = \varphi - \pi - \alpha, \quad \omega_1 = \varphi_1, \quad \alpha < \varphi < 2\pi + \alpha$$
$$\psi_2 = -\varphi - \pi + \alpha, \quad \omega_2 = -\varphi_1, \quad -2\pi + \alpha < \varphi < \alpha$$

$$\rho_k^{-n} e^{in\varphi_k} = \sum_{m=0}^{\infty} (-1)^m C_{m+n-1}^m \rho_j^m R^{-(m+n)} \times$$
$$\times e^{im\varphi_j + i(m+n)\alpha_{kj}}, \quad \rho_j < R; \quad k, j = 1, 2; \quad k \neq j \tag{1.3}$$

We will apply (1.2) with ω_1 and ψ_1 to satisfy the boundary condition on the face $\varphi = \omega > \alpha$ and with ω_2 and ψ_2 on the face $\varphi = 0 < \alpha$.

Formula (1.1) is obtained as follows. The boundary value problem of finding a harmonic function within a circle of radius $\rho_1 < h$ with centre at the point O_1 (Fig.1) is solved. Values of another harmonic function $\rho^{-s}e^{is\varphi}$ are taken as boundary values. Hence we obtain the equality of the two harmonic functions

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$$\rho^{-s}e^{is\varphi} = \sum_{n=0}^{\infty} \rho_1{}^n\,[\bar{a}_n \cos n\varphi_1 + \bar{b}_n \sin n\varphi_1] \quad (\rho_1 < h) \tag{1.4}$$

that holds not only on the boundary but also within a circle. Setting $\varphi_1 = 0$ $(\varphi = \alpha)$ therein, we obtain

$$e^{is\alpha}(\rho_1 + h)^{-s} = \sum_{n=0}^{\infty} \bar{a}_n \rho_1{}^n$$

We will find $\bar{a}_n$ from this equality. Differentiating (1.4) with respect to φ_1 and again setting $\varphi_1 = 0\ (\varphi = \alpha)$ we find $\bar{b}_n$. After elementary reduction we obtain (1.1). The uniqueness of the expansion obtained by this method follows from the uniqueness of the solution of the boundary value problem.

Formula (1.2) can be obtained if the Dirichlet boundary value problem is solved for an angle with apex at the point O containing the ray $\varphi_1 = \pi$.

The equality (1.3) is obtained in an elementary manner from the equations

$$z_k^{-n} = \sum_{m=0}^{\infty} c_m z_j{}^m \quad (k, j = 1, 2;\ k \neq j)$$

$$z_k = x_k + iy_k, \quad R = O_1O_2$$

2. For simplicity, we will consider just one circular hole in an elastic wedge.

Problem 1. Find a harmonic function $u(x, y)$ in a domain Ω that is an angle, with a circle omitted, by means of the boundary conditions

$$u\,|_{\varphi=0} = 0, \quad \partial u/\partial n\,|_{\rho_1=R} = 0 \tag{2.1}$$

$$\partial u/\partial n\,|_{\varphi=\omega} = \mu^{-1}\tau(\rho), \quad \operatorname{grad} u\,|_{\infty} = 0 \tag{2.2}$$

(μ is the shear modulus). We will assume that $|\operatorname{grad} u| \in L_2(\Omega_1)$ where Ω_1 is the neighbourhood of the wedge apex and $\tau(\rho) \in L(0, \infty)$.

We will seek the solution in the form of the expansion /1-4/

$$u = \sum_{n=0}^{\infty} \left(\frac{R}{\rho_1}\right)^n (a_n \cos n\varphi_1 + b_n \sin n\varphi_1) + \tag{2.3}$$

$$\frac{1}{2\pi i} \int_{\Gamma} [A(s) \cos s(\omega - \varphi) + B(s) \sin s\varphi] \frac{\rho^{-s}}{\cos \omega s}\, ds$$

$$(\Gamma\colon\ 0 < \operatorname{Re} s < \delta_1 < 1)$$

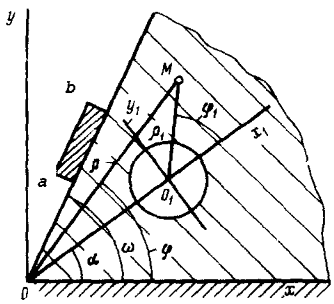

Fig.1

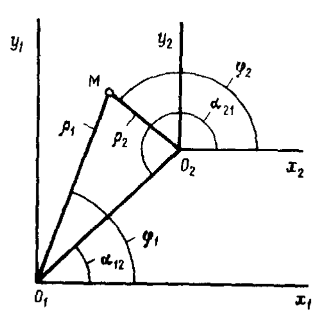

Fig.2

The boundary conditions realized by using expansions (1.1) and (1.2) result in an integro-algebraic system of equations $(a_0 = 0,\ \varepsilon = R/h,\ \beta = \omega - \alpha)$

$$\begin{Vmatrix} a_n \\ b_n \end{Vmatrix} = \frac{\varepsilon^n}{2\pi i n!} \int_{\Gamma} M_n(s) \begin{Vmatrix} \cos \beta s & \sin \alpha s \\ -\sin \beta s & -\cos \alpha s \end{Vmatrix} \begin{Vmatrix} A(s) \\ B(s) \end{Vmatrix} ds \tag{2.4}$$

$$\begin{Vmatrix} A(s) \\ B(s) \end{Vmatrix} = \begin{Vmatrix} 0 \\ \gamma_0 \end{Vmatrix} + \sum_{n=1}^{\infty} \frac{\varepsilon^n \gamma_n(s)}{(n-1)!} \begin{Vmatrix} \cos(\pi - \alpha) s & \sin(\pi - \alpha) s \\ \cos(\pi - \beta) s & \sin(\pi - \beta) s \end{Vmatrix} \begin{Vmatrix} a_n \\ b_n \end{Vmatrix} \tag{2.5}$$

$$M_n(s) = \frac{\Gamma(1-s)\,h^{-s}}{\Gamma(1-s-n)\cos\omega s}, \quad \gamma_n(s) = \frac{\pi\Gamma(s)\,h^{-s}}{\Gamma(1+s-n)\sin\pi s}$$

$$\gamma_0(s) = (\mu s)^{-1}\int_0^\infty \rho^s\tau(\rho)\,d\rho$$

It follows from the geometry of the problem that $0 < \varepsilon < 1$.

We shall temporarily assume that $\tau(\rho)\rho^2 \in L(0, \infty)$ for $0 < \lambda < \delta_1$. It is later possible to get rid of this constraint and to set $\lambda = 0$.

Let us study system (2.4). To do this we introduce the operators

$$\left\|\begin{matrix} D_n^\beta \\ \Gamma_n^\alpha \end{matrix}\right\| f = \frac{\varepsilon^n}{in!}\int_\Gamma M_n(s)\left\|\begin{matrix}\cos\beta s \\ \sin\alpha s\end{matrix}\right\| f(s)\,ds$$

We will examine the operators D_n^a, Γ_n^a $(a = \alpha, \beta)$ in the space $L_2(-\infty, \infty)$.

Theorem 1. The operators D_n^a and Γ_n^a act completely continuously from $L_2(-\infty, \infty)$ to l_2 under the condition

$$\arcsin\varepsilon \leqslant \min(\alpha, \beta) \tag{2.6}$$

We carry out the proof only for D_n^β since it will be analogous for the other operators. We will first show that the series

$$I_1 = \sum_{n=1}^\infty \int_{-\infty}^\infty \left|\frac{\varepsilon^n}{n!} M_n(s) f(s)\cos\beta s\right|^2 d\tau, \quad s = \lambda + i\tau \tag{2.7}$$

converges for any function $f(s) \in L_2(-\infty, \infty)$. We have

$$I_1 = \int_{-\infty}^\infty |f(s)\cos\beta s|^2\,d\tau \sum_{n=1}^\infty \left|\frac{\varepsilon^n}{n!} M_n(s)\right|^2 = \int_{-\infty}^\infty \Big| f(s)\cos\beta s \times$$

$$\frac{h^{-s}}{\cos\omega s}\Big|^2 d\tau\left(\frac{1}{2\pi}|h^s e^{-is\alpha}|^2 \int_0^{2\pi} |\rho^{-s} e^{is\varphi}|^2\,d\varphi_1 - 1\right)$$

The Parseval equality is used here to expand (1.1) in which we have put $\rho_1 = R$. The integral of the second component in parentheses in the last formula converges since $\omega > \beta$. We convert the integral of the first component to the form

$$\frac{1}{2\pi}\int_0^{2\pi} \rho^{-2\lambda}(\varphi_1)\,d\varphi_1 \int_{-\infty}^\infty \left| f(s)\frac{\cos\beta s}{\cos\omega s}\right|^2 e^{2\tau(\alpha-\varphi)}\,d\tau =$$

$$\int_0^{2\pi} \rho^{-2\lambda}(\varphi_1)\,d\varphi_1 \int_{-\infty}^\infty |f(s)|^2 O\left(e^{-2\pi\tau|\alpha + 2\tau(\alpha-\varphi)}\right) d\tau$$

The inner integral obviously converges provided that

$$\arcsin\varepsilon \leqslant \alpha \tag{2.8}$$

This means that the series (2.7) also converges under this condition. If the circle $\rho_1 = R$ intersects the ray $\varphi_1 = 0$, condition (2.8) is violated and the inner integral converges not for all $f(s) \in L_2(-\infty, \infty)$. This means that this condition is not only sufficient for series (2.7) to converge but also necessary.

By using truncated sequences (/5/, Sect.20, par.20.4) the complete continuity of the operator D_n^β from $L_2(-\infty, \infty)$ into l_2 under condition (2.8) is established from the convergence of series (2.7). The theorem is proved for D_n^β. The complete continuity of the remaining operators D_n^α, Γ_n^a $(a = \alpha, \beta)$ from $L_2(-\infty, \infty)$ to l_2 is established by the same method. Condition (2.6) is the general necessary and sufficient condition for complete continuity for the whole set of operators D_n^a and Γ_n^a.

Thus, we have established that a matrix operator is on the right-hand side of (2.4), whose elements are completely continuous operators from $L_2(-\infty, \infty)$ to l_2 under the condition (2.6).

To analyse (2.5) it is convenient to introduce the operators

$$\left\|\begin{matrix} A_\alpha \\ B_\alpha \end{matrix}\right\| \lambda_n = \sum_{n=1}^\infty \frac{\varepsilon^n\gamma_n(s)}{(n-1)!}\left\|\begin{matrix}\cos(\pi-\alpha)s \\ \sin(\pi-\alpha)s\end{matrix}\right\| \lambda_n$$

and to consider them defined in l_2.

Theorem 2. The operators A_a, B_a $(a = \alpha, \beta)$ act completely continuously from l_2 to $L_2(-\infty, \infty)$ under the condition (2.6).

We will prove the theorem for the operator B_β. We will first establish the convergence of the integral

$$I_2 = \int_{-\infty}^{\infty} d\tau \sum_{n=1}^{\infty} \left| \lambda_n \frac{\varepsilon^n \gamma_n(s)}{(n-1)!} \sin s(\pi - \beta) \right|^2 \tag{2.9}$$

for the arbitrary sequence $\lambda_n \in l_2$. We have

$$I_2 = \sum_{n=1}^{\infty} \left| \frac{\varepsilon^n \lambda_n}{(n-1)!} \right|^2 \int_{-\infty}^{\infty} |\gamma_n(s) \sin s(\pi - \beta)|^2 d\tau = \tag{2.10}$$

$$8\pi^3 \int_0^{\infty} \rho^{2\lambda-1} d\rho \sum_{n=1}^{\infty} |\lambda_n|^2 \left(\frac{R}{\rho_1(\rho)}\right)^{2n} \sin^2 n\varphi_1(\rho)$$

The Parseval equality in known form (/6/, p.126, par.3.17), applied to the expansion (1.2) in which we set $\varphi = \omega$ is used in the transformation of the integral (2.10).

Series (2.10) converges under the condition $\min \rho_1 \geqslant R$, which is equivalent to the condition

$$\arcsin \varepsilon \leqslant \beta \tag{2.11}$$

Convergence of the inner series in (2.10) implies the convergence of series (2.9). If condition (2.11) violated (this occurs when the circle $\rho_1 = R$ intersects a side of the angle $\varphi = \omega$) then the inner series in (2.10) will not converge for all $\lambda_n \in l_2$. This means that condition (2.11) is a necessary and sufficient condition for series (2.10) to converge.

We establish from the fact that integral (2.9) converges, by using truncated functions (equal to zero for $|\tau| > l$) in $L_2(-\infty, \infty)$, that the operator B_β acts completely continuously from l_2 into $L_2(-\infty, \infty)$ under condition (2.11), where this condition is necessary and sufficient. The validity of Theorem 2 for A_a, B_α is established by an analogous method. Condition (2.6) is the general necessary and sufficient condition for complete continuity of these operators. The theorem is proved.

Theorem 2 has established that the matrix operator defined by the right-hand side of (2.5) has completely continuous operators from l_2 into $L_2(-\infty, \infty)$ as its elements.

We will also bear in mind that $\gamma_0(s) \in L_2(-\infty, \infty)$ in (2.6) is the condition for the circle not to intersect with the sides of the angle.

3. As a result of eliminating the functions $A(s)$ and $B(s)$ from system (2.4) and (2.5), we arrive at the infinite system of equations

$$\left\| \begin{matrix} a_n \\ b_n \end{matrix} \right\| = \left\| \begin{matrix} \alpha_n \\ \beta_n \end{matrix} \right\| + \sum_{k=1}^{\infty} \left\| \begin{matrix} \alpha_{kn}^{(1)} & \beta_{kn}^{(1)} \\ -\beta_{kn}^{(1)} & -\beta_{kn}^{(2)} \end{matrix} \right\| \left\| \begin{matrix} a^k \\ b_k \end{matrix} \right\| \tag{3.1}$$

$$\left\| \begin{matrix} \alpha_{kn}^{(1)} \\ \beta_{kn}^{(2)} \end{matrix} \right\| = \int_{-\infty}^{\infty} g_{nk}(\tau) \left[\frac{\operatorname{ch} \tau(\pi - \omega)}{\operatorname{sh} \pi\tau} \left\| \begin{matrix} 1 \\ 1 \end{matrix} \right\| + \left\| \begin{matrix} 1 \\ -1 \end{matrix} \right\| \operatorname{sh} \tau(\pi - \alpha) \right] \frac{d\tau}{\operatorname{ch} \tau\omega}$$

$$\beta_{kn}^{(1)} = i \int_{-\infty}^{\infty} g_{nk}(\tau) \frac{\operatorname{ch} \tau(\beta - \alpha)}{\operatorname{ch} \tau\omega} d\tau \tag{3.2}$$

$$g_{nk}(\tau) = \frac{\varepsilon^{n+k}(-1)^n \Gamma(n + i\tau)}{2in!\,(k-1)!\,\Gamma(1 - k + i\tau)}, \quad \left\| \begin{matrix} \alpha_n \\ \beta_n \end{matrix} \right\| = \frac{\varepsilon^n}{\mu n!} \int_{-\infty}^{\infty} \tau(\rho) K_n^{\pm}(\rho)\, d\rho$$

$$K_n^{\pm} = \left\| \begin{matrix} K_n^+ \\ K_n^- \end{matrix} \right\| = \frac{(-1)^n}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma(n + i\tau)}{\Gamma(1 + i\tau)} \left(\frac{\rho}{h}\right)^{i\tau} \left\| \begin{matrix} i \operatorname{sh} \tau\alpha \\ \operatorname{ch} \tau\alpha \end{matrix} \right\| \frac{d\tau}{\operatorname{ch} \tau\omega} \tag{3.3}$$

It turns out to be possible to set $\lambda = 0$ in the system obtained so that the condition on $\tau(\rho)$ appears naturally: $\tau(\rho) \in L(0, \infty)$.

In analysing system (3.1) we start from the fact that its matrix operator is obtained as a result of composition of completely continuous matrix operators. Since the composition of two completely continuous operators is a completely continuous operator, the matrix operator of (3.1) will be completely continuous from l_2 into l_2. We have thereby established the following theorem.

Theorem 3. The operator of system (3.1) is completely continuous in l_2 under condition

(2.6). This condition is also necessary.

Analogous reasoning enables us to deduce that $(\alpha_n, \beta_n) \in l_2$. Therefore, a solution of system (3.1) belonging to l_2 exists, is unique, and can be found by the method of reduction under the condition

$$\arcsin \varepsilon < \min(\alpha, \beta) \tag{3.4}$$

This last assertion follows from the Hilbert alternative /7/ and the uniqueness of the solution of the initial problem of elasticity theory /8/ under the condition that the lines bounding the body do not touch.

Remark. When the wedge is perforated by k circular holes, the solution of the problem should be taken in the form

$$u = \sum_{r=1}^{k} \sum_{n=0}^{\infty} \left(\frac{R_r}{\rho_r}\right)^n (a_n^{(r)} \cos n\varphi_r + b_n^{(r)} \sin n\varphi_r) + u_0$$

where u_0 is the integral component in (2.3). We again obtain a set of k infinite systems which will possess the same property as system (3.1) under the condition that the holes do not touch each other and the sides of the angle, by the method elucidated above and relying on expansion (1.3).

4. We assume in the problem of a stamp that there is no load outside the stamp, while it itself adheres to the elastic body and is shifted along the oz axis by a force T. In this case

$$\gamma_0 = (\mu i\tau)^{-1} \int_a^b \rho^{i\tau} \tau_z(\rho)\, d\rho$$

and the distribution of the tangential forces $\tau_z(\rho)$ must be found from the condition $u|_{\varphi=\omega} = d$ for $a < \rho < b$. The constant d is found from the equilibrium equation

$$\int_a^b \tau_z(\rho)\, d\rho = T$$

From the conditions under the stamp we have the equation

$$\mu^{-1} \int_a^b \tau_z(x) K\left(\ln \frac{x}{\rho}\right) dx = d + \sum_{n=1}^{\infty} \varepsilon^n (a_n H_n^+(\rho) + b_n H_n^-(\rho)) \tag{4.1}$$

$(a < \rho < b)$

$$H_n^{\pm} = \left\| \begin{matrix} H_n^+ \\ H_n^- \end{matrix} \right\| = \frac{1}{(n-1)!} \int_{-\infty}^{\infty} \frac{\Gamma(i\tau)}{\Gamma(1-n-i\tau)} \left(\frac{h}{\rho}\right)^{i\tau} \left\| \begin{matrix} i \operatorname{sh} \tau\alpha \\ \operatorname{ch} \tau\alpha \end{matrix} \right\| \frac{d\tau}{\operatorname{ch} \tau\omega} \tag{4.2}$$

$$K(z) = \frac{1}{\pi} \int_0^{\infty} \tau^{-1} \operatorname{th} \tau\omega \cos \tau z\, d\tau$$

We make the following substitutions in (4.1)

$$\xi = a^* \ln x + b^*, \quad t = a^* \ln \rho + b^*$$

$$a^* = \frac{2}{\ln(b/a)}, \quad b^* = \frac{\ln(ab)}{\ln(a/b)}$$

$$\frac{x}{a^*} \tau_z(x) = \varphi(\xi), \quad \varphi(\xi) = \sum_{k=0}^{\infty} \lambda_k T_k(\xi)(1-\xi^2)^{-1/2}$$

where $T_k(x)$ are Chebyshev polynomials, and we extract the logarithmic term from the series $K(z)$:

$$K(z) = -\frac{1}{\pi} \ln|z| + F(z),$$

$$F(z) = \int_0^{\infty} \tau^{-1} [e^{-\tau} + (\operatorname{th} \tau\omega - 1) \cos \tau z]\, d\tau$$

Applying the procedure of the method of orthogonal polynomials /9/ to the equation obtained, we arrive at the infinite system

$$\lambda_0 \delta = \sum_{k=0}^{\infty} \lambda_k K_{0k} - \mu \sum_{k=0}^{\infty} \varepsilon^k A_{k0} \quad (\delta = -\pi \ln 2|a^*|) \tag{4.3}$$

$$\lambda_n = \frac{2n}{\pi}\left(\mu\sum_{k=1}^{\infty}\varepsilon^k A_{kn} - \sum_{k=0}^{\infty}\lambda_k K_{nk}\right)$$

$$K_{nk} = I_t^{(n)} I_\xi^{(k)} F\left(\frac{\xi - t}{a^*}\right), \quad I_x^{(n)} f = \int_{-1}^{1} f(x)\frac{T_n(x)}{\sqrt{1-x^2}}\,dx \tag{4.4}$$

$$A_{kn} = a_k M_{kn}^+ + b_k M_{kn}^-, \quad M_{kn}^{\pm} = I_t^{(n)} H_k^{\pm}(\rho), \quad \rho = \exp\frac{t-b^*}{a^*}$$

We append the system

$$\begin{Vmatrix} a_n \\ b_n \end{Vmatrix} = \sum_{k=1}^{\infty}\begin{Vmatrix} \alpha_{kn}^{(1)} & \beta_{kn}^{(1)} \\ -\beta_{kn}^{(1)} & -\beta_{kn}^{(2)} \end{Vmatrix}\begin{Vmatrix} a_k \\ b_k \end{Vmatrix} + \sum_{k=0}^{\infty}\lambda_k L_{kn}^{\pm} \tag{4.5}$$

obtained from (3.1) as a result of substituting the expansion for $\tau_z(\rho)$ therein, to the above system. The matrix elements have the form (3.2) and

$$L_{kn}^{\pm} = \begin{Vmatrix} L_{kn}^+ \\ L_{kn}^- \end{Vmatrix} = \frac{\varepsilon^n}{\mu n!}\begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix} I_t^{(k)} K_n^{\pm}\left(\exp\frac{t-b^*}{a^*}\right)$$

The set of sysstems (4.3) and (4.5), together with the statics condition, forms a closed system of equations to determine the unknowns a_n, b_n, λ_n and d.

5. Let us investigate system (4.3) and (4.5). We note first that the series $\sum_{k,n} n^2 |K_{nk}|^2$ converges. This follows from the fact that the function $F(z)$ has derivatives of any order. Let us prove the convergence of the series $\sum_{k,n} |L_{kn}^{\pm}|^2$, $\sum_{k,n} \varepsilon^{2k} n^2 |M_{kn}^{\pm}|^2$. We introduce the function

$$\varphi(\eta, \rho) = \sum_{k=1}^{\infty}\eta^k \frac{d}{d\rho} H_k^+(\rho), \quad \eta = \varepsilon e^{i\theta} \tag{5.1}$$

(θ is a real number). We will show that this is an analytic function of the variable η in a certain circle. Taking account of (4.4) and (4.2), we have $(y = h/\rho)$

$$\varphi(\eta, \rho) = -i\rho\sum_{k=1}^{\infty}\frac{\eta^k}{(k-1)!}\int_0^{\infty}\frac{\Gamma(1+i\tau)\,y^{i\tau}}{\Gamma(1-k-i\tau)}\frac{\operatorname{sh}\tau\alpha}{\operatorname{sh}\tau\omega}\,d\tau =$$

$$i\rho\sum_{k=1}^{\infty}\frac{(-\eta)^k}{(k-1)!}\int_{-\infty}^{\infty}\frac{y^{i\tau}\operatorname{sh}\tau\alpha}{\Gamma(-i\tau)\operatorname{ch}\tau\omega}\,d\tau\int_0^{\infty}e^{-t}t^{k+i\tau-1}\,dt =$$

$$-i\frac{\eta\rho}{1+\eta}\int_{-\infty}^{\infty}\left(\frac{y}{1+\eta}\right)^{i\tau}\frac{\Gamma(1+i\tau)\operatorname{sh}\tau\alpha}{\Gamma(-i\tau)\operatorname{ch}\tau\omega}\,d\tau$$

Formulas 8.334 (3), 8.310 (1) and 3.381 (4) from /10/ are used successively here.

In last integral converges and has a derivative with respect to η provided that $\arcsin|\eta| = \arcsin\varepsilon < \beta = \omega - \alpha$ where this condition is necessary and sufficient for the integral to converge. Therefore, the function $\varphi(\eta, \rho)$ will be analytic in the circle $|\eta| < \sin\beta$ for $\beta \leqslant \pi/2$ and in the circle $|\eta| < 1$ for $\beta > \pi/2$. The inequalities

$$\left|\frac{d}{d\rho} H_k^+(\rho)\right| \leqslant \frac{C}{(\sin\beta)^k}, \quad k = 1, 2, \ldots; \quad C = \text{const} \tag{5.2}$$

follow from the Cauchy inequalities for the coefficients of the series (5.1) for $|\eta| = \varepsilon < \sin\beta$ and $\beta \leqslant \pi/2$. For $\beta > \pi/2$ the $\sin\beta$ in (5.2) should be replaced by unity. The same estimate holds for the function $H_k^-(\rho)$.

Let us investigate the convergence of the series

$$G = \sum_{n,k=1}^{\infty} n^2\varepsilon^{2k}|M_{kn}^+|^2 = \sum_{k=1}^{\infty}\varepsilon^{2k}\sum_{n=1}^{\infty}|nI_t^{(n)}H_k^+|^2 = \tag{5.3}$$

$$\frac{2}{\pi}\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}}\sum_{k=1}^{\infty}\varepsilon^{2k}\left|\frac{d}{dt}H_k^+(\rho)\right|^2, \quad \rho = \exp\frac{t-b^*}{a^*}$$

The Parseval equality was used here to expand the function $dH_k^+(\rho)/dt$ in a series of Chebyshev polynomials $T_k(t)$. Taking account of the estimate (5.2) we will have the following inequality for series (5.3):

$$G = c_1 \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} \sum_{k=1}^{\infty} \left(\frac{\varepsilon}{\sin \beta}\right)^{2k}, \quad c_1 = \text{const} \tag{5.4}$$

Since inequality (5.2) holds under the condition $\sin \beta > \varepsilon$ when $\beta \leqslant \pi/2$, series (5.4) also converges under this condition. For $\beta > \pi/2$ the $\sin \beta$ in (5.4) should be replaced by unitv. The series G will also converge in this case since $\varepsilon < 1$. Therefore, the condition $\arcsin \varepsilon < \beta$ will be not only the sufficient but also the necessary conditon for series (5.3) to converge. Convergence of the series $\sum_{k,n} |L_{kn}^{\pm}|^2$ is proved analogously. The convergence of the double series of the squares of the moduli of the matrix coefficients $\alpha_{kn}^{(1)}, \beta_{kn}^{(1,2)}$ was proved earlier under condition (3.4). Complete continuity of the matrix operators of systems (4.3) and (4.5) (/5/,p.216) in l_2 follows from the convergence of the series noted.

We have thereby established the following theorem.

Theorem 4. Condition (3.4) is necessary and sufficient for the complete continuity of matrix operators of the right-hand sides of systems (4.3) and (4.5) in the space l_2.

An approximate solution of the infinite systems of problems 1 and 2 can be obtained by the method of reduction or in the form of expansions in the small parameter ε.

The case of several circular holes can be examined in an analogous manner.

REFERENCES

1. LIN C.-B. and HSU C.-M., Stresses in a perforated wedge, Trans. ASME, Ser. E, J. Appl. Mech., 40, 3, 1973.
2. PROTSENKO V.S. and SOLOV'YEV A.I., On the joint application of Cartesian and bipolar coordinates to the solution of boundary value problems of potential theory and elasticity thoery, PMM, 48, 6, 1984.
3. BABLOYAN A.A. and TADEVOSYAN R.G., The problem for a composite plane weakened by a polygonal hole, Izv. Akad. Nauk Armssr, Mekhanika, 39, 1986.
4. PROTSENKO V.S., SOLOV'YEV A.I. and TSYMBALYUK V.V., Torsion of elastic bodies constrained by coordinate surfaces of toroidal and spherical coordinate systems, PMM, 50, 3, 1986.
5. TRENOGIN V.A., Functional Analysis. Nauka, Moscow, 1980.
6. TITCHMARSH E., Introduction to the Theory of Fourier Integrals, Gostekhizdat, Moscow-Leningrad, 1948.
7. KANTOROVICH L.V. and KRYLOV V.I., Approximate Methods of Higher Analysis, Fizmatgiz, Moscow-Leningrad, 1962.
8. BURCHULADZE T.V. and GEGELIYA T.G., Development of the Potential Method in Elasticity Theory. Metsniereba, Tbilisi, 1985.
9. POPOV G.YA., Elastic Stress Concentration Near Stamps, Slits, Thin Inclusions, and Reinforcements. Nauka, Moscow, 1982.
10. GRADSHTEIN I.S. and RYZHIK I.M., Tables of Integrals, Sums, Series and Products. Fizmatgiz, Moscow, 1962.

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